Gagliardo-Nirenberg pointwise estimate via sparse domination method

Tomáš Roskovec

University of South Bohemia in České Budějovice
E-mail: tomas.roskovec@gmail.com

Estimating the intermediate derivative of a function by the product of powers of function and higher derivative can be done in several ways. The most important is the original normwise estimate in the form

$$\|D^j u\|_p \leq C\|D^k u\|_q^{\frac{j}{k}} \|u\|_r^{1-\frac{j}{k}}, \quad \text{where } 1 \leq j < k,$$

degrees of derivatives, coefficients of Lebesgue spaces, and powers are fixed by additional conditions. We focus on the pointwise estimate of Kałamajska–Mazya–Shaposhnikova type

$$|D^j u| \leq C(MD^k u)^{\frac{j}{k}} (Mu)^{1-\frac{j}{k}}, \quad \text{where } 1 \leq j < k.$$

Such an inequality is extremely beneficial, as it allows the development of the GN inequalities for general Banach functional spaces and replicates the original result with a very short proof for most Lebesgue cases but fails in spaces where the maximal operator is not bounded, such as L^1 (spaces with the upper Boyd index equal to 1).

Our contribution is overcoming the limitation of the KMS inequality by replacing the maximal operator with a special averaging method called sparse domination. Such an operator is bounded even for spaces uncovered by previous methods and enriches the family of proven GN inequalities.